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Chapter 1

The quadrilateral

1.1 General notions

In this section we give some general properties of the quadrilateral.

Definition 1.1.1. Let A, B, C, D be four points with the property that no three of them are collinear, $[AB] \cap [CD] = \emptyset$ and $[BC] \cap [DA] = \emptyset$. Then the set $Q = [AB] \cup [BC] \cup [CD] \cup [DA]$ is called a quadrilateral. The points A, B, C, D are the vertices, $[AB]$, $[BC]$, $[CD]$, $[DA]$ are the sides and $[AC]$, $[BD]$ are the diagonals of the quadrilateral. Note by A, B, C, D the measures of the angles $\widehat{DAB}$, $\widehat{ABC}$, $\widehat{BCD}$, $\widehat{CDA}$ of the quadrilateral (Fig. 1.1.1)

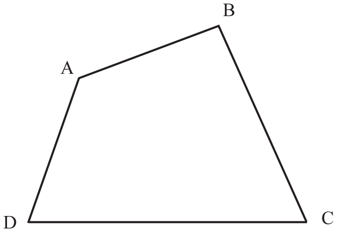

Fig. 1.1.1

Remark 1.1.1. The shapes in Fig. 1.1.2 are not quadrilaterals.

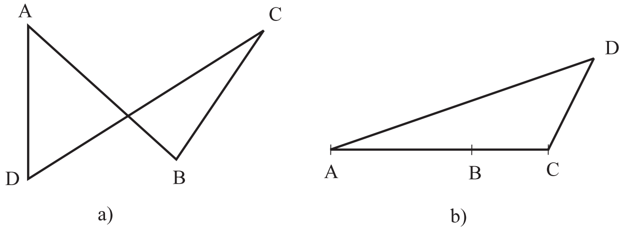

Fig. 1.1.2

Definition 1.1.2. The quadrilateral $ABCD$ is called convex if, for any side, the vertices which don't lie on it, are on the same half-plane determined by the support of the side (Fig. 1.1.3).

Definition 1.1.3. A quadrilateral is called concave if it isn't convex (Fig. 1.1.4).

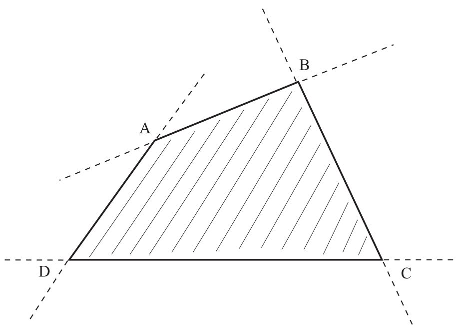

Fig. 1.1.3

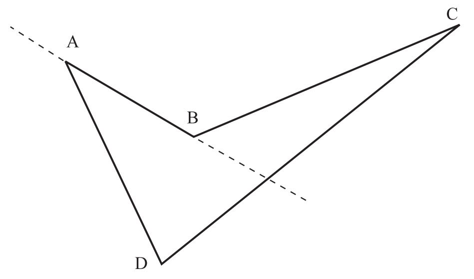

Fig. 1.1.4

Definition 1.1.4. The interior of a convex quadrilateral is the intersection of the open half-planes which are limited by the supports of the sides of the quadrilateral and contain the vertices that are not situated on their limit (Fig. 1.1.3).
We note the interior of the convex quadrilateral $ABCD$ by $(ABCD)$.

Definition 1.1.5. The union of a convex quadrilateral and its interior is called a convex quadrilateral surface. The convex quadrilateral surface of the convex quadrilateral $ABCD$ is noted $[ABCD]$ and we have that $[ABCD] = Q \cup (ABCD)$, where $Q = [AB] \cup [BC] \cup [CD] \cup [DA]$.

1.2 Determining conditions

Theorem 1.2.1. *Let a, b, c, d be strictly positive, real numbers. These numbers can be the lengths of the sides of a quadrilateral if and only if $a < b + c + d$, $b < c + d + a$, $c < d + a + b$ and $d < a + b + c$.*

Proof. Let $ABCD$ be a quadrilateral with the lengths of the sides a, b, c, d (Fig. 1.2.1). In the triangles ABC and ACD we have that $a + b > AC$, and $AC + c > d$ respectively, whence $a + b + c > d$. Similarly the other inequalities are demonstrated.

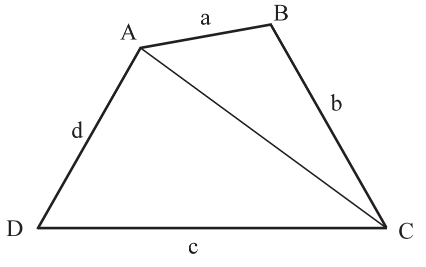

Fig. 1.2.1

Now, we prove the converse statement. If the inequalities $a+b>c+d$, $b+c>d+a$, $c+d>a+b$ and $d+a>b+c$ hold simultaneously, then, summing them, we obtain that $0>0$, which is a contradiction.

So, there exists an inequality which doesn't take place, say $a+b\leq c+d$.

Let $\varepsilon_1=a$, $\varepsilon_2=b$, $\varepsilon_3=a+b-c+d$, $\varepsilon_4=a+b+c-d$, $\varepsilon=\min(\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4)$ and $a'=a+b-\dfrac{\varepsilon}{2}$. Taking $\varepsilon\leq\varepsilon_3$ into account, it results that $a'=a+b-\dfrac{\varepsilon}{2}\geq a+b-\dfrac{a+b-c+d}{2}=\dfrac{a+b+c-d}{2}>0$, so $a'>0$. Because $a'\geq\dfrac{a+b+c-d}{2}$ and $d+a+b>c$, we have that $a'+d\geq\dfrac{a+b+c-d}{2}+d=\dfrac{(d+a+b)+c}{2}>c$, so $a'+d>c$.

Similarly it can be demonstrated that $a'+c>d$. But $a'=a+b-\dfrac{\varepsilon}{2}<a+b\leq c+d$, so we have proved that $a'<c+d$, $c<a'+d$ and $d<a'+c$. Then, it results that a',c,d can be the lengths of the sides of a triangle ACD, where $CD=c$, $DA=d$ and $AC=a'$. Taking $\varepsilon\leq a$ into account, we have that $a'+a=\left(a+b-\dfrac{\varepsilon}{2}\right)+a\geq\dfrac{3a}{2}+b>b$, so $a'+a>b$. Similarly one has that $a'+b>a$.

But $a'=a+b-\dfrac{\varepsilon}{2}<a+b$, so we have proved that $a'<a+b$, $a<a'+b$ and $b<a'+a$, which means that a',a,b can be the lengths of the sides of a triangle ABC, where $AB=a$ and $BC=b$.

In conclusion, we prove that there exists a quadrilateral $ABCD$, with sides $AB=a$, $BC=b$, $CD=c$ and $DA=d$. □

Theorem 1.2.2. *Let a,b,c,d be strictly positive real numbers. These numbers can be the lengths of the sides of a quadrilateral if and only if strictly positive real numbers x,y,z,t exist such that:* $a=\dfrac{-x+y+z+t}{2}$, $b=\dfrac{x-y+z+t}{2}$, $c=\dfrac{x+y-z+t}{2}$ *and* $d=\dfrac{x+y+z-t}{2}$.

Proof. If a,b,c,d are the lengths of the sides of a quadrilateral, then we consider the set of equations

$$\frac{-x+y+z+t}{2}=a, \frac{x-y+z+t}{2}=b, \frac{x+y-z+t}{2}=c, \frac{x+y+z-t}{2}=d,$$

whence $x+y+z+t=2s$, $x=s-a$, $y=s-b$, $z=s-c$, $t=s-d$, where $s=\dfrac{a+b+c+d}{2}$. But, taking Theorem 1.2.1 into account, we have $x=s-a=\dfrac{-a+b+c+d}{2}>0$ and the analogous inequalities.

Reciprocally, we calculate $a+b+c-d=\dfrac{-x+y+z+t}{2}+\dfrac{x-y+z+t}{2}+\dfrac{x+y-z+t}{2}-\dfrac{x+y+z-t}{2}=2t>0$, so $a+b+c>d$ and its analogues. Then, taking Theorem 1.2.1 into account, it results that a,b,c,d can be the lengths of the sides of a quadrilateral. □

Corollary 1.2.1. *In the conditions of Theorem* 1.2.2*, we have*

$$\sum a=\sum x, \tag{1.2.1}$$

$$\sum ab=\sum xy, \tag{1.2.2}$$

$$\sum a^2=\sum x^2 \tag{1.2.3}$$

and

$$(s-a)(s-b)(s-c)(s-d)=xyzt. \tag{1.2.4}$$

Proof. It results through calculations. □

Theorem 1.2.3 (D. Pompeiu's Theorem)**.** *Let $ABCD$ be a parallelogram and M a point in the plane different from the points A,B,C and D. Prove that there exists a quadrilateral with sides of lengths MA, MB, MC and MD respectively.*

Proof. We will prove that $MD<MA+MB+MC$. The other inequalities can be proved similarly.

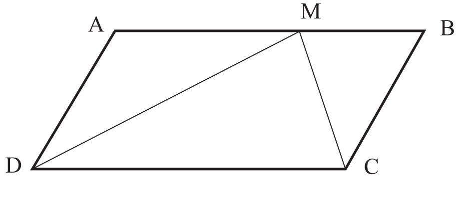

Fig. 1.2.2

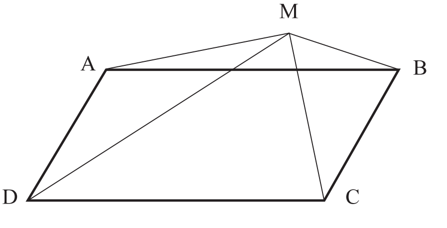

Fig. 1.2.3

If $M \in (AB)$ (Fig. 1.2.2), then $AB = MA + MB$ and $MA + MB + MC = AB + MC = DC + MC$. In triangle DMC we have $DC + MC > MD$ and then it results that $MA+MB+MC > MD$. If $M \notin (AB)$ (Fig. 1.2.3), then $AB < MA + MB$ and $MA + MB + MC > AB + MC = DC + MC$.
In triangle MDC, which can be degenerate, we have $DC + MC \geq MC$ and in this case we also obtain the desired inequality. □

Theorem 1.2.4. *Let a, b, c and d be strictly positive numbers. The following statements are equivalent:*

(*i*) *there exists a quadrilateral with sides of lengths a, b, c and d respectively;*

(*ii*) *there exists a convex quadrilateral with side lengths a, b, c and d respectively.*

Proof. If (ii) holds, then (i) also holds. Suppose that there exists a concave quadrilateral $ABCD$ with side lengths a, b, c and d (Fig. 1.2.4), where $C \in Int \triangle ABD$. Let C' be the symmetric of C with respect to the midpoint of BD. Then $ABC'D$ is a convex quadrilateral with sides of lengths a, b, c and d. □

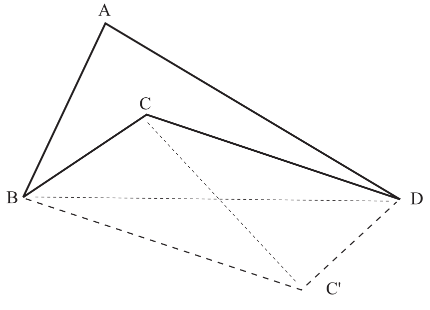

Fig. 1.2.4

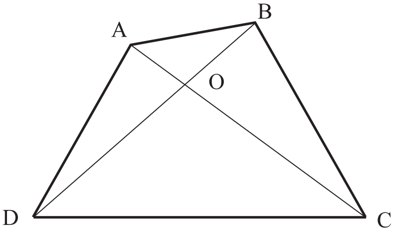

Fig. 1.2.5

Theorem 1.2.5. *A quadrilateral $ABCD$ is convex if and only if $(AC) \cap (BD) \neq \emptyset$.*

Proof. If $ABCD$ is a convex quadrilateral, then $A \in \text{Int}\,\widehat{BCD}$ (Fig. 1.2.5) and because $B \in (CB$, $D \in (CD$, it results that $(CA \cap (BD) \neq \emptyset$.
Similarly we have that $(AC) \cap (DB \neq \emptyset$. From the relations above, it results that $(AC) \cap (BD) \neq \emptyset$.

Conversely, if for the quadrilateral $ABCD$ we have that $(AC) \cap (BD) \neq \emptyset$, let $(AC) \cap (BD) = \{O\}$. From $(AC) \cap (BD) = \{O\}$, it results that the points B and D are situated in different half-planes determined by the line AC. Similarly, the points A and C are situated in different half-planes determined by the line BD. Because $(BO) \cap DC = \emptyset$ and $(AO) \cap DC = \emptyset$, it results that the points A, O and B are situated on the same side of the line DC. Similarly, we have that the points A, O, D are situated on the same side of the line BC, the points C, O, D are situated on the same side of the line AB and the points B, O, C are situated on the same side of the line AD. From the remarks above, it results that $ABCD$ is a convex quadrilateral. □

Theorem 1.2.6. *A quadrilateral is convex if and only if the sum of the measures of its angles equals 2π.*

Proof. If the quadrilateral is convex, considering a point in its interior, it results immediately that the sum of the measures of its angles equals 2π. Reciprocally, assume that the quadrilateral $ABCD$ is not convex and that the points B and C are situated in opposite half-planes determined by the line AD (Fig. 1.2.6). We have that $AD \cap (BC) = \{X\}$.

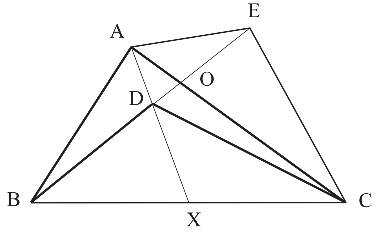

Fig. 1.2.6

Then $D \in \text{Int}\,\widehat{ABC}$. Let E be a point in the half-plane determined by the line AC, opposed to the one in which point B lies. One verifies immediately that $ABCE$ is a convex quadrilateral. From the triangles ADE and DEC we have that

$$\widehat{ADC} = 2\pi - (\widehat{EAD} + \widehat{AEC} + \widehat{ECD}).$$

If the sum of the measures of the angles of quadrilateral $ABCD$ equals 2π, then $\widehat{DAB} + \widehat{ABC} + \widehat{BCD} + \widehat{CDA} = 2\pi$. From the last two equalities we have that $\widehat{DAB} + \widehat{ABC} + \widehat{BCD} - (\widehat{EAD} + \widehat{AEC} + \widehat{ECD}) = 0$, which leads to $\widehat{EAB} + \widehat{ABC} + \widehat{BCE} + \widehat{CEA} = 2(\widehat{EAD} + \widehat{AEC} + \widehat{ECD})$. Since $ABCE$ is a convex quadrilateral, $\widehat{EAB} + \widehat{ABC} + \widehat{BCE} + \widehat{CEA} = 2\pi$, and so $\widehat{EAD} + \widehat{AEC} + \widehat{ECD} = \pi$. Replacing this in the first equality, we obtain $\widehat{ADC} = \pi$, so the points A, D and C are collinear. But this is in contradiction with the fact that $ABCD$ is a quadrilateral. □

1.3 Euler's Theorem and Leibniz-type relations

In the following, let E and F be the midpoints of diagonals AC and BD of quadrilateral $ABCD$ and let G be the midpoint of EF.

Theorem 1.3.1 (Euler's Theorem)**.** *In a quadrilateral $ABCD$, the following identity*

$$4EF^2 = AB^2 + BC^2 + CD^2 + DA^2 - AC^2 - BD^2 \tag{1.3.1}$$

holds.

Proof. Expressing the median length in triangles ADC, ABC and DEB (Fig. 1.3.1),

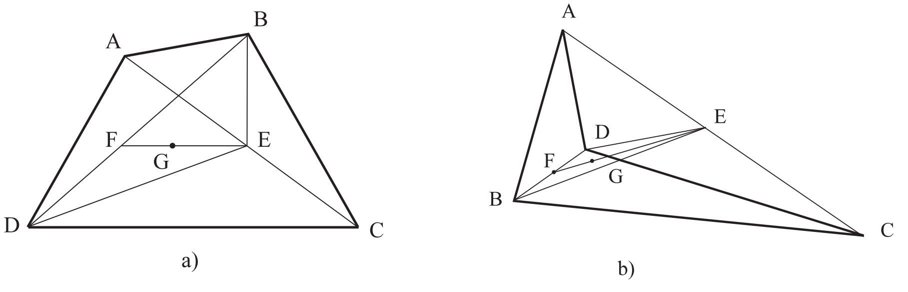

Fig. 1.3.1

we have that $DE^2 = \frac{DA^2 + DC^2}{2} - \frac{AC^2}{4}$, $BE^2 = \frac{BA^2 + BC^2}{2} - \frac{AC^2}{4}$ and $EF^2 = \frac{EB^2 + ED^2}{2} - \frac{BD^2}{4}$. By replacing the expressions of DE^2 and BE^2 in that of EF^2, we obtain (1.3.1). □

Theorem 1.3.2. *Let $ABCD$ be a quadrilateral. The following affirmations are equivalent:*

(i) *the quadrilateral $ABCD$ is a parallelogram;*

(ii) *the relation $AB^2 + BC^2 + CD^2 + DA^2 = AC^2 + BD^2$ holds.*

Proof. If the quadrilateral $ABCD$ is a parallelogram, then the midpoints E and F of diagonals AC and BD coincide, so from Euler's relation, (ii) results.

Reciprocally, from (ii) and Euler's relation it results that E and F coincide, so the diagonals AC and BD of the quadrilateral $ABCD$ meet at their common midpoint. Therefore, $ABCD$ is a parallelogram. □

From the relation (1.3.1) and Theorem 1.3.2, the following corollary results.

Corollary 1.3.1. *In a quadrilateral $ABCD$ we have the inequality*

$$AB^2 + BC^2 + CD^2 + DA^2 \geq AC^2 + BD^2. \tag{1.3.2}$$

Equality holds if and only if $ABCD$ is a parallelogram.

Lemma 1.3.1. *Let $ABCD$ be a quadrilateral and M, N the midpoints of the sides AD and BC respectively. Then $MN \leq \frac{AB + CD}{2}$ and the equality holds if and only if $AB \| DC$.*

Proof. Because EM and EN are medians (Fig. 1.3.2), $EM = \dfrac{CD}{2}$, $EM \| CD$, $EN = \dfrac{AB}{2}$ and $EN \| AB$.

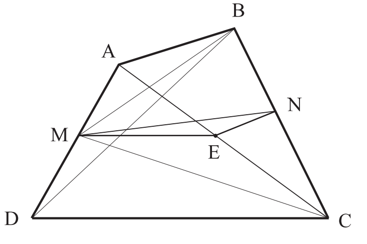

Fig. 1.3.2

In the triangle MEN, which can be a degenerate triangle, we have that $MN \leq EM + EN$, thus the conclusion. The equality holds if and only if $E \in MN$, or equivalently $AB \| MN$ and $DC \| MN$, which means $AB \| DC$.

□

Lemma 1.3.2. *Let $ABCD$ be a quadrilateral. If M, N are the midpoints of sides AD and BC respectively, then*

$$4MN^2 = AB^2 - BC^2 + CD^2 - DA^2 + AC^2 + BD^2. \tag{1.3.3}$$

Proof. Applying the median theorem in triangles MCB, ACD and ABD (Fig. 1.3.2), we have that

$$\begin{aligned} 4MN^2 &= 2(MC^2 + MB^2) - BC^2 \\ &= \left(CD^2 + CA^2 - \frac{AD^2}{2}\right) + \left(BD^2 + BA^2 - \frac{AD^2}{2}\right) - BC^2, \end{aligned}$$

which yields (1.3.3). □

Theorem 1.3.3. *Let $ABCD$ be a quadrilateral. Then*

$$AC^2 + BD^2 \leq AD^2 + BC^2 + 2AB \cdot CD, \tag{1.3.4}$$

and the equality holds if and only if $AB \| CD$.

Proof. From Lemma 1.3.1 and Lemma 1.3.2 we have that

$$AB^2 - BC^2 + CD^2 - DA^2 + AC^2 + BD^2 = 4MN^2 \leq 4\left(\frac{AB + CD}{2}\right)^2,$$

which gives us (1.3.4). □

Remark 1.3.1. From Theorem 1.3.3 it results that if $ABCD$ is a trapezoid so that $AB \| CD$, then $AC^2 + BD^2 = AD^2 + BC^2 + 2AB \cdot CD$.

Lemma 1.3.3. *In the quadrilateral $ABCD$, the identity*

$$GA^2 + GB^2 + GC^2 + GD^2 = \frac{AC^2 + BD^2}{2} + EF^2 \qquad (1.3.5)$$

holds, where E, F and G are the midpoints of AC, BD and EF.

Proof. Considering the length of the medians in triangles AGC and BGD (Fig. 1.3.3), we have that

$$GE^2 = \frac{2(GA^2 + GC^2) - AC^2}{4} \quad \text{and} \quad GF^2 = \frac{2(GB^2 + GD^2) - BD^2}{4}.$$

Summing the last two relations, (1.3.5) follows. □

Theorem 1.3.4 (Leibniz's relation)**.** *If X is an arbitrary point in the plane of the quadrilateral $ABCD$, then*

$$XA^2 + XB^2 + XC^2 + XD^2 = 4XG^2 + GA^2 + GB^2 + GC^2 + GD^2. \quad (1.3.6)$$

Proof. In the triangles XAC, XBD and XEF (Fig. 1.3.3), we have that

$$XE^2 = \frac{2(XA^2 + XC^2) - AC^2}{4}, \; XF^2 = \frac{2(XB^2 + XD^2) - BD^2}{4}$$

and $XG^2 = \dfrac{2(XE^2 + XF^2) - EF^2}{4}$.

By replacing the expressions of XE^2 and XF^2 in that of XG^2 and taking Lemma 1.3.3 into account, we obtain relation (1.3.6).

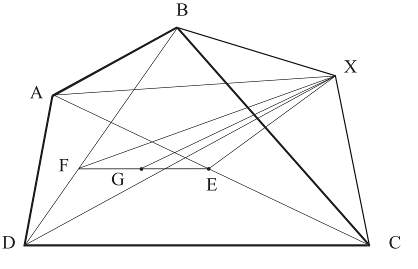

Fig. 1.3.3

□

Corollary 1.3.2. *If X is an arbitrary point in the plane of quadrilateral $ABCD$, then*

$$XA^2+XB^2+XC^2+XD^2-4XG^2=\frac{1}{4}\,(a^2+b^2+c^2+d^2+e^2+f^2) \qquad (1.3.7)$$

and

$$XA^2 + XB^2 + XC^2 + XD^2 \geq \frac{1}{4}\,(a^2 + b^2 + c^2 + d^2 + e^2 + f^2), \qquad (1.3.8)$$

so the sum $XA^2 + XB^2 + XC^2 + XD^2$ is minimal if and only if X coincides with G.

Proof. The identity (1.3.7) follows from the relations (1.3.1), (1.3.5) and (1.3.6). From (1.3.7), (1.3.8) is obtained. □